Simulation

Random-Variate Generation

Dr. Xueping Li
University of Tennessee

Based on
Banks, Carson, Nelson & Nicol
Discrete-Event System Simulation
Purpose & Overview

- Develop understanding of generating samples from a specified distribution as input to a simulation model.

- Illustrate some widely-used techniques for generating random variates.
  - Inverse-transform technique
  - Acceptance-rejection technique
  - Special properties
Inverse-transform Technique

The concept:

- For cdf function: \( r = F(x) \)
- Generate \( r \) from uniform \((0,1)\)
- Find \( x \):

\[
x = F^{-1}(r)
\]
Exponential Distribution

- Exponential Distribution:
  - Exponential cdf:
    
    \[ r = F(x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0 \]

  - To generate \( X_1, X_2, X_3 \ldots \)

    \[ X_i = F^{-1}(R_i) = -(1/\lambda) \ln(1-R_i) \]

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Figure: Inverse-transform technique for \( \exp(\lambda = 1) \)
Example: Generate 200 variates $X_i$ with distribution $\exp(\lambda = 1)$

- Generate 200 $R$s with $U(0,1)$ and utilize equation from the previous slide, the histogram of $X$s become:

- Check: Does the random variable $X_1$ have the desired distribution?

$$P(X_1 \leq x_0) = P(R_1 \leq F(x_0)) = F(x_0)$$
Other Distributions

Examples of other distributions for which inverse cdf works are:

- Uniform distribution
- Weibull distribution
- Triangular distribution
Empirical Continuous Dist’n [Inverse-transform]

- When theoretical distribution is not applicable
- To collect empirical data:
  - Resample the observed data
  - Interpolate between observed data points to fill in the gaps
- For a small sample set (size $n$):
  - Arrange the data from smallest to largest
    \[ x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)} \]
  - Assign the probability $1/n$ to each interval \( x_{(i-1)} \leq x \leq x_{(i)} \)

\[
X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i \left( R - \frac{(i-1)}{n} \right)
\]

where \( a_i = \frac{x_{(i)} - x_{(i-1)}}{1/n - (i-1)/n} = \frac{x_{(i)} - x_{(i-1)}}{1/n} \)
Empirical Continuous Dist’n

[Inverse-transform]

Example: Suppose the data collected for 100 broken-widget repair times are:

<table>
<thead>
<tr>
<th>i</th>
<th>Interval (Hours)</th>
<th>Frequency</th>
<th>Relative Frequency</th>
<th>Cumulative Frequency, c_i</th>
<th>Slope, a_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25 ≤ x ≤ 0.5</td>
<td>31</td>
<td>0.31</td>
<td>0.31</td>
<td>0.81</td>
</tr>
<tr>
<td>2</td>
<td>0.5 ≤ x ≤ 1.0</td>
<td>10</td>
<td>0.10</td>
<td>0.41</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>1.0 ≤ x ≤ 1.5</td>
<td>25</td>
<td>0.25</td>
<td>0.66</td>
<td>2.0</td>
</tr>
<tr>
<td>4</td>
<td>1.5 ≤ x ≤ 2.0</td>
<td>34</td>
<td>0.34</td>
<td>1.00</td>
<td>1.47</td>
</tr>
</tbody>
</table>

Consider \( R_1 = 0.83 \):

\( c_3 = 0.66 < R_1 < c_4 = 1.00 \)

\( X_1 = x_{(4-1)} + a_4(R_1 - c_{(4-1)}) \)

\( = 1.5 + 1.47(0.83-0.66) \)

\( = 1.75 \)
All discrete distributions can be generated via inverse-transform technique.

Method: numerically, table-lookup procedure, algebraically, or a formula.

Examples of application:
- Empirical
- Discrete uniform
- Gamma
Example: Suppose the number of shipments, \( x \), on the loading dock of IHW company is either 0, 1, or 2

- **Data - Probability distribution:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( p(x) )</th>
<th>( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>1</td>
<td>0.30</td>
<td>0.80</td>
</tr>
<tr>
<td>2</td>
<td>0.20</td>
<td>1.00</td>
</tr>
</tbody>
</table>

- **Method - Given \( R \), the generation scheme becomes:**

\[
x = \begin{cases} 
0, & R \leq 0.5 \\
1, & 0.5 < R \leq 0.8 \\
2, & 0.8 < R \leq 1.0 
\end{cases}
\]

Consider \( R_1 = 0.73 \):
- \( F(x_{i-1}) < R \leq F(x_i) \)
- \( F(x_0) < 0.73 \leq F(x_1) \)
Hence, \( x_1 = 1 \)
Acceptance-Rejection technique

- Useful particularly when inverse cdf does not exist in closed form, a.k.a. thinning
- Illustration: To generate random variates, $X \sim U(1/4, 1)$

Procedures:
Step 1. Generate $R \sim U[0,1]$
Step 2a. If $R \geq \frac{1}{4}$, accept $X=R$.
Step 2b. If $R < \frac{1}{4}$, reject $R$, return to Step 1

- $R$ does not have the desired distribution, but $R$ conditioned ($R'$) on the event $\{R \geq \frac{1}{4}\}$ does.
- Efficiency: Depends heavily on the ability to minimize the number of rejections.
NSPP  [Acceptance-Rejection]

- Non-stationary Poisson Process (NSPP): a Possion arrival process with an arrival rate that varies with time

- Idea behind thinning:
  - Generate a stationary Poisson arrival process at the fastest rate, \( \lambda^* = \max \lambda(t) \)
  - But “accept” only a portion of arrivals, thinning out just enough to get the desired time-varying rate

```
Generate E \sim \text{Exp}(\lambda^*)
\text{t} = \text{t} + E
```

```
Condition
R \leq \lambda(t)
```

```
Output E' \sim t
```

Diagram:
- Generate E \sim \text{Exp}(\lambda^*)
- t = t + E
- Condition R \leq \lambda(t)
- Output E' \sim t
Example: Generate a random variate for a NSPP

Data: Arrival Rates

<table>
<thead>
<tr>
<th>t (min)</th>
<th>Mean Time Between Arrivals (min)</th>
<th>Arrival Rate λ(t) (#/min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>15</td>
<td>1/15</td>
</tr>
<tr>
<td>60</td>
<td>12</td>
<td>1/12</td>
</tr>
<tr>
<td>120</td>
<td>7</td>
<td>1/7</td>
</tr>
<tr>
<td>180</td>
<td>5</td>
<td>1/5</td>
</tr>
<tr>
<td>240</td>
<td>8</td>
<td>1/8</td>
</tr>
<tr>
<td>300</td>
<td>10</td>
<td>1/10</td>
</tr>
<tr>
<td>360</td>
<td>15</td>
<td>1/15</td>
</tr>
<tr>
<td>420</td>
<td>20</td>
<td>1/20</td>
</tr>
<tr>
<td>480</td>
<td>20</td>
<td>1/20</td>
</tr>
</tbody>
</table>

Procedures:

Step 1. \( \lambda^* = \max \lambda(t) = 1/5, \ t = 0 \) and \( i = 1 \).

Step 2. For random number \( R = 0.2130 \),

\[
E = -5 \ln(0.213) = 13.13
\]

\( t = 13.13 \)

Step 3. Generate \( R = 0.8830 \)

\[
\lambda(13.13)/\lambda^* = (1/15)/(1/5) = 1/3
\]

Since \( R > 1/3 \), do not generate the arrival

Step 2. For random number \( R = 0.5530 \),

\[
E = -5 \ln(0.553) = 2.96
\]

\( t = 13.13 + 2.96 = 16.09 \)

Step 3. Generate \( R = 0.0240 \)

\[
\lambda(16.09)/\lambda^* = (1/15)/(1/5) = 1/3
\]

Since \( R < 1/3 \), \( T_1 = t = 16.09 \),

and \( i = i + 1 = 2 \)
Special Properties

- Based on features of particular family of probability distributions

- For example:
  - Direct Transformation for normal and lognormal distributions
  - Convolution
  - Beta distribution (from gamma distribution)
Direct Transformation

Approach for normal(0,1):

- Consider two standard normal random variables, $Z_1$ and $Z_2$, plotted as a point in the plane:

  In polar coordinates:
  \[
  Z_1 = B \cos \phi \\
  Z_2 = B \sin \phi
  \]

- $B^2 = Z_1^2 + Z_2^2 \sim \text{chi-square distribution with 2 degrees of freedom} = \text{Exp}(\lambda = 2)$. Hence, $B = (-2 \ln R)^{1/2}$

- The radius $B$ and angle $\phi$ are mutually independent.

\[
\begin{align*}
Z_1 &= (-2 \ln R)^{1/2} \cos(2\pi R) \\
Z_2 &= (-2 \ln R)^{1/2} \sin(2\pi R)
\end{align*}
\]
Direct Transformation

- **Approach for normal** $(\mu, \sigma^2)$:
  - Generate $Z_i \sim N(0, 1)$

\[
X_i = \mu + \sigma Z_i
\]

- **Approach for lognormal** $(\mu, \sigma^2)$:
  - Generate $X \sim N((\mu, \sigma^2)$

\[
Y_i = e^{X_i}
\]
Summary

- Principles of random-variate generate via
  - Inverse-transform technique
  - Acceptance-rejection technique
  - Special properties

- Important for generating continuous and discrete distributions